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Particle Systems Acting on Undirected Graphs

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We study a class of interacting particle systems where the states of two neighboring sites are simultaneously changed in dependence on their previous states. For two states, this class is parametrized by six parameters. Duality and existence of stationary distributions is discussed.

KEY WORDS: Interacting particle system; cellular automaton; graph; dual process.

1. INTRODUCTION

Cellular automata and interacting particle systems have much in common. Usually they act on regular grids like $\mathbb{Z}^d$, use 0, 1 states, and the transition mechanism concerns the change of the state at a single site, depending on the states of sites in a neighborhood. Generally the mathematical investigation of these systems follows different lines, in cellular automata the dynamical systems view and combinatorial aspects dominate, interacting particle systems are stochastic processes. The theory of interacting particle systems has been established in a very general framework,[12] classes of examples where detailed information can be obtained are far more special.[5, 6, 12, 13]

Cellular automata can be used to model various local reactions between "species" but it is difficult to model directly spatial effects like migration or diffusion. Therefore in ref. 14 a class of "dimer automata" has been introduced in which, in contrast to classical cellular automata, the states of two neighboring sites are changed such that the new states depend only on the previous states at these two sites. There are some examples of

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interacting particle systems where the states of two sites are interchanged or changed at the same time such as the exclusion process[12] and annihilating processes.[7] For two elementary states 0, 1 there are 256 dimer automata. In ref. 14 these were enumerated (following the enumeration of Wolfram[15] for cellular automata), their equivalence classes under a group of elementary transformations were determined and the asymptotic behavior explored by mean field approximations and computer simulations.

In ref. 2 it was observed that the concept of a dimer automaton, originally designed for $\mathbb{Z}$ with a three-site neighborhood, can be carried over to arbitrary graphs: an edge of the graph is called at random, and new states are attributed to the two adjacent vertices according to a deterministic function. These processes were called "edge processes." Many features of such systems, as for instance the existence of a dual (see Section 3) or of a stationary product measure (see Section 5), do not strongly depend on the graph but only on the local rule.

Here we generalize the concept of an edge process in such a way that, once an edge is called, new states are selected for the two adjacent vertices according to some random distribution with the previous two states at these sites as parameters. Our motivation to study such processes comes on one hand from quasiperiodic and random tilings, which are used as models of quasicrystals and more disordered materials,[1,11] and on the other hand from modelling in the biological and social sciences where quite irregular graphs are used to describe relations between individuals or groups, and where particle systems are likely to be more frequently applied in the near future.[8]

Thus transition rates will be defined for the edges of the graph, and the states at the two vertices of an edge may change simultaneously. Our class of edge processes includes many well-known spin systems with linear rates, the deterministic dimer automata as well as the exclusion process. Nevertheless, this class is sufficiently small to be studied as a whole. For our case of two elementary states 0, 1 and undirected graphs, it can be parametrized by points in a six-dimensional simplex.

This parametrization appears natural: to a certain extent, properties of the processes can be described using the geometry of the parameter set; the properties of a convex combination reflect those of its constituents. Several interesting subclasses correspond to convex subsets of the parameter simplex and it seems possible to investigate two- and three-parameter families of processes in a similar way as this is done for chaotic dynamical systems. In particular we show that only additive edge processes[10] have a dual, and we perform a mean-field calculation to find stationary distributions. For a subset of codimension 1, it can be rigorously proved that a stationary product measure exists.

2. EDGE PROCESSES

Our approach is implicitly contained in classical work of Harris[9, 10] (cf. Durett[5]), and formulated in cellular automata language in ref. 14.

Definition. An *edge process* is given by

1. a finite set F of elementary states,

2. a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with countably many vertices and the additional property that no edge connects a vertex to itself and that at each vertex there are only finitely many ingoing or outgoing edges,

3. a map $Q: F^2 \times F^2 \to [0, \infty)$.

The vertices of the graph can be seen as sites and the edges as defining a neighborhood structure for these sites. The classical example is $\mathbf{V} = \mathbb{Z}^d$ with the Moore or the von Neumann neighborhood structure. The elementary states can be seen as colors, their number will be denoted by m. In the simplest case we have $F = \{0, 1\}$ or the colors white and black. The configuration space is $F^{\mathbf{V}} = \{\xi: \mathbf{V} \to F\}$. Thus a configuration associates a color to each site. The state of the site itself is $\xi(x)$. The map Q is a matrix $Q(u, v)$ with m^2 rows and columns.

The edge process works as follows. At an event a directed edge (rather than a site) is selected, the states u_1 at the outgoing site and u_2 at the ingoing site are inspected, collected into the ordered pair $u = (u_1, u_2)$ and two new values $v = (v_1, v_2)$ are attributed to these sites. The rates $Q(u, v)$ define when and how these events take place. Thus an edge with states u has exponentially distributed holding time with respect to a change into v with parameter $\lambda = Q(u, v)$ unless another event changes either u_1 or u_2, i.e., u into u' from whereon the parameter is $Q(u', v)$. To avoid unnecessary changes from u to u, we assume $Q(u, u) = 0$ for all u.

Before we introduce other descriptions of the transition mechanism, we briefly recall the concept of *equality of processes*. Two particle systems are equal (in distribution) if they behave in the same way. This means that for each fixed initial configuration, the distribution of the configuration at any time t (and also the joint distribution at times $t_1,..., t_n$) is the same for both systems. In the following two edge processes will also be considered to be equivalent if one is obtained from the other by a uniform time change with a factor r. That is, the distribution of $(\xi(t_1),..., \xi(t_n))$ for the first process agrees with the distribution of $(\xi(rt_1),..., \xi(rt_n))$ for the second, for every choice of $t_1,..., t_n$. In other words, two matrices Q, Q' will describe the same process if $Q = rQ'$. Thus we shall assume throughout that $\sum_{u, v \in F^2} Q(u, v) = 1$.

2.1. Transitions Simplified

So far we assumed that each transition $u \mapsto v$ is performed with rate $Q(u, v)$. With the above convention on Q, we now reformulate the transition mechanism so that the $Q(u, v)$ appear as transition probabilities for pairs of states. Each edge e is called with rate 1, and if the states at the two sites of e are given by u, they are changed to the new pair of states v with probability $Q(u, v)$. (Thus even if the edge is called, there will be no change with probability $1 - \sum_{\{v \mid v \neq u\}} Q(u, v)$.) A standard argument concerning thinning of Poisson processes shows that this mechanism agrees with the previous one (cf. ref. 6).

For finite graphs $\mathbf{G}$, the long-time behaviour of the process is not changed if we work with discrete time, choosing at each step one of the edges with respect to the equidistribution on $\mathbf{E}$ and applying Q as above. This modified edge process is in fact an ordinary Markov chain.

2.2. Comparison with Spin Systems

Some edge processes can be interpreted as particle systems with flip rates $c_i(x, \xi)$ for the change $\xi(x) \mapsto i$ at *single sites* x. The necessary and sufficient condition for Q is $Q(u, v) = 0$ whenever $u_1 \neq v_1$ and $u_2 \neq v_2$. When this condition is fulfilled, $c_i(x, \xi)$ can be calculated as a sum which depends *linearly* on the numbers $k_j^+(x, \xi)$ and $k_j(x, \xi)$ of neighbours of x with state j via outgoing and incoming edges, respectively. Conversely, if a particle system is given by flip rates $c_i(x, \xi) = \sum_{j \in F} a_{ilj} k_j^+ + b_{ilj} k_j^-$ for $\xi(x) = l \neq i$ where a_{ilj}, b_{ilj} are nonnegative numbers, it can be represented as an edge process. (The proof given in Proposition 2.2 below easily extends to the general case.)

Thus the local rules in our approach are rather special. On the other hand, edge processes include most of the familiar examples in refs. 5, 6, 12, and 13 as for instance voter models, contact processes, and exclusion processes. Moreover, our approach is general in the sense that it applies to arbitrary directed graphs $\mathbf{G}$, not only to regular lattices as $\mathbb{Z}^d$. Even for $\mathbf{V} = \mathbb{Z}^d$, edges can be chosen to define different types of neighbourhoods (Moore and von Neumann nearest-neighbour, long-range, or asymmetric neighbourhoods for asymmetric exclusion processes). It seems an advantage of our definition that transition mechanism Q and graph structure are separated.

2.3. The Family of Edge Processes

Initially we have introduced edge processes by their rates $Q(u, v)$. Thus the family of edge processes is parametrized by the nonnegative matrices

$Q = (Q(u, v))$ of order m^2. The zero matrix is the trivial process which changes nothing. Then we have, without lack of generality, assumed

$$Q(u, u) = 0, \qquad \sum_{u, v} Q(u, v) = 1 \tag{$*$}$$

This normalization excludes the trivial process. Whenever we need not specify the underlying graph, we shall describe the process by the matrix.

An edge process is called deterministic if there is $c > 0$ such that all $Q(u, v)$ are either zero or c, and for each $u \in F^2$ there is at most one v with $Q(u, v) = c$. The constant c is determined by ($*$). For $m = 2$ we have the 256 deterministic edge processes (including the trivial process) studied in ref. 14. The normalization ($*$) determines a simplex $\mathscr{F}$. Its vertices are the matrices with ($*$) which have exactly one element equal to 1. These $m^4 - m^2$ matrices (12 for $m = 2$) are deterministic processes. They can be conveniently described by their unique nontrivial transition $u \mapsto v$ (cf. Table 1). The simplex $\mathscr{F}$ is the convex hull of these deterministic processes.

The description of edge processes by a convex compact parameter set which is the convex hull of deterministic processes is not only formal. If an edge process is a convex combination of certain deterministic processes (in the parameter space) then one can interpret the action as calling the actions of the deterministic edge processes according to the probability distribution given by the convex combination.

2.4. Undirected Graphs

In the following we concentrate on $F = \{0, 1\}$, and we study only undirected graphs, that is, directed graphs such that (x, y) is an edge if and

Table 1. The Extreme Points of $\mathscr{F}$ for Undirected Graphs

Name	Description	Rule
A	Annihilation	$11 \mapsto 00$
B	spontaneous Birth	$00 \mapsto 01$
C	Coalescence	$11 \mapsto 01$
D	Dying out	$01 \mapsto 00$
E	Exclusion process	$01 \mapsto 10$
T	spontaneous Twin birth	$00 \mapsto 11$
G	Richardson Growth model	$01 \mapsto 11$

only if (y, x) is an edge. An undirected edge $\{x, y\}$ is defined as the union of the directed edges (x, y) and (y, x), and the selection of directed edges works as above. For regular graphs of degree k, the selection of (x, y) can also be described as follows[5,6]: first x is chosen, applying an exponentially distributed holding time to each point, and then one of the k neighbours is selected using probabilities $1/k$. On undirected graphs, the two processes $10 \mapsto v_1 v_2$ and $01 \mapsto v_2 v_1$ are equal. Moreover, the processes $11 \mapsto 10$ and $11 \mapsto 01$ coincide: If $\{x, y\}$ is an edge and $\xi(x) = \xi(y) = 1$, both processes transform ξ to ξ' with $\xi'(x) = 0$, $\xi'(y) = 1$ with rate 1, and to ξ'' with $\xi''(x) = 1$, $\xi''(y) = 0$ with the same rate. The same holds for $00 \mapsto 10$ and $00 \mapsto 01$. Thus the number of extreme points of $\mathscr{F}$ will decrease from 12 to 7 for undirected graphs and these are the 7 rules given in Table 1. Since an edge is a pair of directed edges, the processes A, B, C, and T act with rate 2. Thus we have shown the following result.

Proposition 2.1. The space $\mathscr{F}$ of all edge processes for $F = \{0, 1\}$ and undirected graphs is the 6-dimensional simplex generated by the seven deterministic processes listed in Table 1.

In other words, the transition matrix Q of an edge process has a representation $Q = aA + bB + cC + dD + eE + tT + gG$ with non-negative numbers fulfilling $a + b + c + d + e + t + g = 1$. The terminology in Table 1 uses the interpretation of state 0 as "empty" or "dead" site and 1 as "occupied" or "living." Then the configuration ξ can be identified with the set $\{x \mid \xi(x) = 1\}$, and $(\xi_t)_{t \geqslant 0}$ can be considered as a *set-valued Markov process.*[9,10,12] An important subclass of processes is given by the condition that "nothing can develop from 0" which we call "legal" processes (following a similar definition of Wolfram[15] for cellular automata). This subclass excludes B and T and is thus represented by the 4-dimensional subsimplex of $\mathscr{F}$ with vertices A, C, D, E, and G. An interacting particle system over $\{0, 1\}$ is called a spin system[5,12] if only one site can change at a time. This excludes A, E, and T.

Proposition 2.2. A spin system can be represented as an edge process if and only if the flip rates at a site x depend linearly on the number $k_j = k_j(x, \xi)$ of neighbours which have state j. All linear spin systems form a 3-dimensional subsimplex of $\mathscr{F}$ with vertices B, C, D, and G.

Proof. For $\xi(x) = 0$, the change $0 \mapsto 1$ will occur with rate $c_1(x, \xi) = bk_0 + gk_1$. For $\xi(x) = 1$, the state of x flips with rate $c_0(x, \xi) = dk_0 + ck_1$. Clearly, all positive linear functions c_0 and c_1 can be realized by suitable choice of b, c, d, g. ∎

3. ADDITIVE PROCESSES AND DUALITY

3.1. Additive Processes

Harris[10] introduced the important class of *additive* particle systems by requiring that

$$\xi_t^{R \cup S} = \xi_t^R \cup \xi_t^S \qquad \text{for all } t \geqslant 0 \text{ and all sets } R, S \subseteq \mathbf{V}$$

Here $R = \xi_0^R$ denotes the starting configuration of ξ_t^R and equality should hold for appropriate realizations of all processes ξ_t^R, $R \subseteq \mathbf{V}$. Harris proved that an edge process is additive if it is given by a 2×2-matrix M in the following way: if the edge (x, y) is called then $(\xi(x), \xi(y))$ is replaced by $(\xi'(x), \xi'(y) = (\xi(x), \xi(y))\, M$. Matrix multiplication is as usual, with the exception $(1, 1)\binom{1}{1} = 1$. Table 2 lists all these edge processes. Harris proved that the additive edge processes coincide with the convex hull of these matrix processes. Since the first five processes in Table 2 are extreme points of the convex hull, and the last two are not, this implies the following statement.

Proposition 3.1. The additive edge processes, considered as subset of $\mathscr{F}$, form the polyhedron with vertices D', D'', E, G, V, W.

3.2. Duality

Harris introduced additive processes to prove the existence of duals. The set-valued process $(\eta_t)_{t \geqslant 0}$ is called a dual of $(\xi_t)_{t \geqslant 0}$ if for all sets of sites $R, S \subseteq \mathbf{V}$ and all $t \geqslant 0$

$$P[\xi_t^R \cap S = \varnothing] = P[\eta_t^S \cap R = \varnothing]$$

Table 2. Additive Edge Processes and Their Matrices

Shorthand	Description	Generating matrices
$D' = (C + D)/2$	spin system $1 \mapsto 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$D'' = (A + 2D)/3$	fast extinction	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
E	exclusion process	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
G	growth model	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$V = (D + G)/2$	voter model	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$
$W = (C + E)/2$	coalescing random walk	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$
$(C + D + E)/3$	(not extreme)	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$(E + G)/2$	(not extreme)	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Duality is a very important concept for the study of particle systems.[5, 6, 12] Putting $R = \varnothing$, we see that only legal processes can have duals. Harris proved that an additive process has a dual, given by the transpose of the matrices.[10] He gave a simple example of an edge process on a directed graph which has a dual and is not additive. We show that this cannot happen for undirected graphs.

Theorem 3.2. Let (ξ_t) be a legal edge process with parameters a, c, d, e, $g \geqslant 0$, acting on an undirected graph $\mathbf{G}$ with at least one edge. Then the following conditions are equivalent.

(i) (ξ_t) is additive

(ii) (ξ_t) has a dual

(iii) The following inequalities hold:

$$d \geqslant 2a \qquad d \geqslant 2a + c - e \qquad d \leqslant 2a + c + g$$

Proof. Harris proved that (i) implies (ii), we show that (ii) implies (iii). Take an edge $\{x, y\}$ from $\mathbf{G}$ and suppose that there are k edges from x to other points and l edges from y, including $\{x, y\}$. All our starting configurations will fulfil $\xi_0(z) = 0$ for $z \notin \{x, y\}$, and only small t will be studied.

If $\xi_0 = \{x\}$ then a first change can occur only at one of the k directed edges (z, x) with $z \neq x$, more precisely,

$$P[\xi_t^{\{x\}} \cap \{x\} = \varnothing] = k(d+e)\, t + o(t) \qquad \text{with} \quad o(t)/t \to 0 \qquad \text{for} \quad t \to 0$$

Any dual process (η_t) is necessarily legal, as shown above, and thus is given by parameters a', c', d', e', g', and $P[\eta_t^{\{x\}} \cap \{x\} = \varnothing] = k(d' + e')\, t + o(t)$. Now apply the definition of duality with $R = S = \{x\}$ to find $d + e = d' + e'$.

Next apply a similar argument to $R = \{x\}$, $S = \{y\}$,

$$P[\xi_t^{\{x\}} \cap \{y\} \neq \varnothing] = (e + g)\, t + o(t)$$

$$P[\eta_t^{\{y\}} \cap \{x\} \neq \varnothing] = (e' + g')\, t + o(t)$$

to get $e + g = e' + g'$. Then take $R = S = \{x, y\}$,

$$P[\xi_t^{\{x, y\}} \cap \{x, y\} = \varnothing] = 2at + o(t)$$

and similarly for η_t. It follows that $a = a'$.

Finally, choose $R = \{x\}$ and $S = \{x, y\}$,

$$P[\xi_t^{\{x\}} \cap \{x, y\} = \varnothing] = (kd + (k-1)\,e)\,t + o(t)$$

$$P[\eta_t^{\{x, y\}} \cap \{x\} = \varnothing] = ((k-1)(d' + e') + 2a' + c')\,t + o(t)$$

Using $d + e = d' + e'$ we find $d = 2a' + c'$. Interchanging R and S we get $d' = 2a + c$. So far we have shown

Proposition 3.3. If the legal edge process (ξ_t) has a dual (η_t), then the parameters a', c', d', e', g' of (η_t) are uniquely determined by the conditions

$$a' = a, \qquad c' = d - 2a, \qquad d' = 2a + c$$

$$e' = d + e - d' = d + e - 2a - c$$

$$g' = e + g - e' = g - d + 2a + c$$

Proposition 3.4. In the parameter space a, c, d, e, $g \geqslant 0$, $a + c + d + e + g = 1$ the inequalities (iii) define a convex polyhedron $\mathscr{C}$ with the following structure. $\mathscr{C}_0 = \mathscr{C} \cap \{d = 2a + c\}$ is a 3-simplex with vertices D', D'', E, G. The set $\mathscr{C}_+ = \mathscr{C} \cap \{d \geqslant 2a + c\}$ is a 4-simplex spanned by $\mathscr{C}_0$ and V, and $\mathscr{C}_- = \mathscr{C} \cap \{d \leqslant 2a + c\}$ is a 4-simplex spanned by $\mathscr{C}_0$ and W.
The transition 3.3 from (ξ_t) to (η_t) carries $\mathscr{C}_+$ into $\mathscr{C}_-$ and vice versa, and leaves $\mathscr{C}_0$ invariant. In particular V and W are dual processes, and $\mathscr{C}_0$ is the set of self-dual processes.

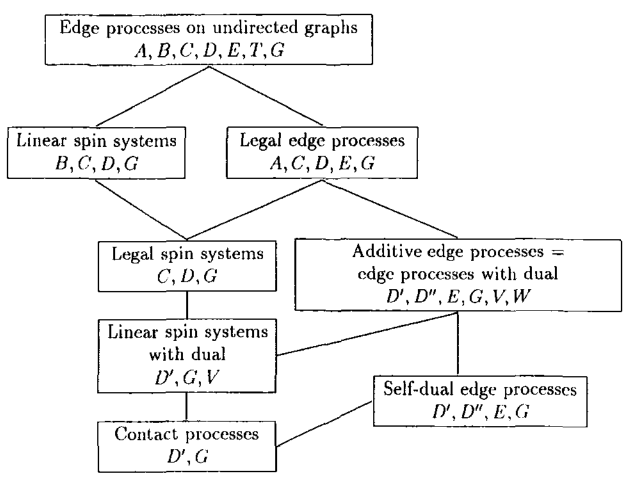

Fig. 1. Classes of edge processes and their extreme points.

Proof of Proposition 3.4. The legal edge processes form a 4-simplex $\mathscr{S}=\{a,\ c,\ d,\ e,\ g\geqslant 0,\ a+c+d+e+g=1\}$ in $\mathbb{R}^5$, with vertices $A=(1,0,0,0,0)$, $C, D, E, G=(0,0,0,0,1)$.

We first assume $d=2a+c$. Then the inequalities (iii) are satisfied so that $\mathscr{C}_0=\mathscr{S}\cap\{d=2a+c\}$. We have $a, c, e, g\geqslant 0$ and $3a+2c+e+g=1$ so that we can calculate the extreme points by setting $3a=1$; $2c=1$; $e=1$ and $g=1$, obtaining $D'=(1/3, 0, 2/3, 0, 0)$, $D''=(0, 1/2, 1/2, 0, 0)$, E and G, respectively. Moreover, by 3.3, $d=2a+c$ is equivalent to $d=d'$ and implies $(\xi_t)=(\eta_t)$ so that $\mathscr{C}_0$ really contains the self-dual processes.

Next, we consider $\mathscr{C}_+=\mathscr{C}\cap\{d\geqslant 2a+c\}$. Put $\delta=d-2a-c$. Then $a, c, e, g, \delta\geqslant 0$, $3a+2c+e+g+\delta=1$, and (iii) reduces to the condition $\delta=g$. As before, the extreme points D', D'', E, G of $\mathscr{C}_+$ are found by putting $3a=1$ etc. Instead of $\delta=1$ we obtain $\delta=g=1/2$ which in old coordinates is $V=(0, 0, 1/2, 0, 1/2)$.

For $\mathscr{C}_-=\mathscr{C}\cap\{d\leqslant 2a+c\}$ we put $\sigma=d-2a$ and $\tau=2a+c-d$ so that $a, e, g, \sigma, \tau\geqslant 0$ and $3a+e+g+2\sigma+\tau=1$. The condition (iii) reduces to $\tau\leqslant e$. As before, the only new vertex corresponds to $\tau=e=1/2$, that is, $W=(0, 1/2, 0, 1/2, 0)$ in old coordinates. The remaining part of 3.4 follows from 3.3. ∎

Figures 2 and 3 show the intersection of $\mathscr{S}$ (dotted lines) and $\mathscr{C}$ with the hyperplanes $a=0$ and $a=1/6$. The partition $\mathscr{C}=\mathscr{C}_+\cup\mathscr{C}_0\cup\mathscr{C}_-$ is visible. Proposition 3.3 says that the duality transformation $\xi(t)\mapsto\eta(t)$ is represented in parameter space by the affine reflection f at the hyperplane

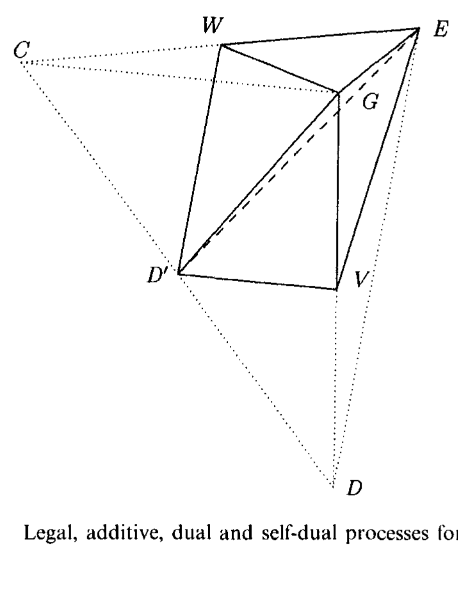

Fig. 2. Legal, additive, dual and self-dual processes for $a=0$.

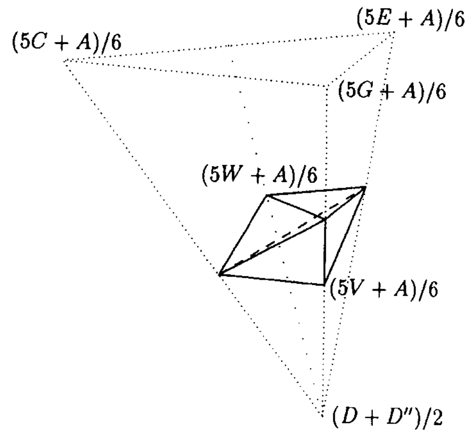

Fig. 3. Legal, additive, dual and self-dual processes for $a = 1/6$.

$\{d = 2a + c\}$ with $f(V) = W$. The proof of Theorem 3.2 implies that $\mathscr{C} = \mathscr{S} \cap f(\mathscr{S})$ which is also indicated by the figures. Moreover, Fig. 3 shows that for larger a we have less processes with dual; for $a = 1/3$, the only remaining process with dual is D'', the bottom vertex of the corresponding tetrahedron.—The legal linear spin systems form a 2-simplex with vertices C, D, G which appears as left front face in Fig. 2. The linear spin systems with dual form the 2-subsimplex spanned by D', G, V. The self-dual processes among them form a 1-simplex spanned by D', G. These are well-known as contact processes.[5, 9, 12]

4. MEAN-FIELD APPROXIMATION

Stationary distributions on the configuration space play a central role in the theory of particle systems. In particular, one wants to know whether the system is ergodic, i.e., whether it converges to a unique stationary distribution for any initial configuration.

We may assume that the graph **G** is connected.

For *finite* **G**, we can ask whether the Markov chain with state space $\{0, 1\}^{\mathbf{V}}$ is irreducible and non-periodic. This question is not difficult to answer but quite a number of particular cases have to be considered.[2] Here we confine ourselves to cases where $a, b, c, d > 0$ so that each configuration can be transformed into any other one with positive probability. Moreover, the presence of a 2-cycle $00 \mapsto 01 \mapsto 00$ and a 3-cycle $00 \mapsto 11 \mapsto 01 \mapsto 00$ shows that the Markov chain is non-periodic.

Proposition 4.1. On a finite connected graph, all edge processes with $a, b, c, d > 0$ represent irreducible and non-periodic Markov chains. In particular, all processes in the interior of the simplex $\mathscr{F}$ of Proposition 2.1 are ergodic.

In the case of infinite graphs, even for graphs with a simple structure, the problem is much harder.[13] It is possible to identify processes which are trivially ergodic in the sense that the all-zero configuration defines the only stationary distribution. However, a given process may die out on one graph $\mathbf{G}$ and survive on the other. Even for $\mathbf{G} = \mathbb{Z}$ there are open problems. For example, the exact critical value s^* of the one-dimensional contact process $(1-s)\,D' + s \cdot G$ is still unknown.[12,13] Moreover, recent results on the contact process on homogeneous trees show that a process may die out on each fixed finite subset of $\mathbf{V}$ and may still survive globally on $\mathbf{V}$.[13] Finally, if a process is not trivially ergodic, it is not easy to determine the stationary distributions and their domains of attraction.

Mean-field approximation is an attempt to find stationary distributions of an edge process, or at least an asymptotic density p_1 of 1's, without reference to the underlying graph $\mathbf{G}$. This method is not rigorous: we make the very restrictive assumption that there is a stationary distribution of the edge process which is a product measure $\mu = \{p_0, p_1\}^{\mathbf{V}}$ on the configuration space $\{0, 1\}^{\mathbf{V}}$. Under this assumption we shall determine p_1 uniquely and show that μ is *stable* under perturbations within the set of product measures.

The assumption approximately holds for a *complete* graph with many vertices ($|\mathbf{V}| \to \infty$), for hydrodynamic limits with "rapid stirring"[4,6] (in our context this means processes with $e \approx 1$ on large graphs $\mathbf{G}$), and for long-range interactions (ref. 6, Chapter 7). We show in the next section that for many edge processes there are indeed stationary product measures for the action on arbitrary graphs. In any case, the mean-field calculation gives a first approximation. For the deterministic case, the coincidence with numerical simulations is remarkable, with few exceptions.[14]

We write $p_1 = p$ and $p_0 = 1 - p$. Then the mean field assumption says that an edge is marked 00 with probability $(1-p)^2$ and 11 with p^2. In the steady state, the rate of change of p_1 is zero:

$$(1-p)^2 \cdot 2(b+2t) + 2p(1-p)(g-d) - p^2 \cdot 2(c+2a) = 0$$

Notice that the parameter e enters implicitly as $e = 1 - a - b - c - d - g - t$. Define the quadratic function

$$f(p) := p^2(b + 2t + d - g - c - 2a) + p(g - d - 2b - 4t) + b + 2t$$

A value p with $f(p)=0$ indicates a stationary distribution and the sign of $f'(p)$ indicates its stability (see below).

Proposition 4.2. Except for the voter model combined with an exclusion process, the mean-field approximation always yields a unique stable probability $p_1 = p^*$. Moreover, $0 < p^* < 1$ except for the growing and terminating processes described below.

Proof. Observe $f(0) = b + 2t \geqslant 0$ and $f(1) = -c - 2a \leqslant 0$.

Case 1. $f(0) > 0$, $f(1) < 0$. Since f has at most two zeros, there is exactly one zero $p^* \in (0, 1)$, and $f'(p) < 0$. For $0 < p_1 < p^*$ we have $f(p) > 0$ so that in the average, new 1's are created, while for $p^* < p_1 < 1$ we have $f(p) < 0$ so that 1's are removed.

Case 2. $f(0) = 0$ (i.e., the process is legal) and $f(1) < 0$. Then consider $f'(0) = g - d$. If $g > d$ then $f'(0) > 0$ implies again that there is a unique $p^* \in (0, 1)$, $f'(p^*) < 0$, which is stable, attracting all $p_1 \neq 0$. If $g \leqslant d$, then $p^* = 0$ is the only root, and $f(p) < 0$ for $0 < p \leqslant 1$. Such processes will be called *terminating*.

Case 3. $f(1) = 0$, $f(0) > 0$. Then $f'(1) = d - g$. For $g < d$ there is a unique $p^* \in (0, 1)$, p^* is stable. For $g \geqslant d$ we have $p^* = 1$, $f(p) > 0$ for $0 \leqslant p < 1$. Such processes will be called *growing*.

Case 4. $f(0) = f(1) = 0$. Then $b = t = c = a = 0$, $f(p) = p(1 - p(g - d)$. The process is a linear combination of D, E, G. Thus the process is a *biased voter model*[5, 6] combined with an exclusion process. For $g > d$ the process is growing, $p^* = 1$ is stable. For $g < d$, $p^* = 0$ is stable, and the process is terminating. The singular case $g = d$, $f(p) \equiv 0$ represents the basic *voter model*, combined with an exclusion process. For this case it is known that on certain graphs, both $p^* = 0$ and $p^* = 1$ may be the result of the process while on other graphs like $\mathbb{Z}^3$, there are steady states with arbitrary p_1.[5, 12]

The mean field density has itself some convexity property.

Proposition 4.3. Let Q and $\bar{Q}$ be edge processes with mean-field probabilities p^* and $\bar{p}^*$, and let $\tilde{Q} = sQ + (1 - s)\bar{Q}$ be a convex combination. Then $\tilde{p}^*$ lies between p^* and $\bar{p}^*$.

Proof. The quadratic function $\tilde{f}(p)$ is also a convex combination $\tilde{f}(p) = sf(p) + (1 - s)\bar{f}(p)$. If $p^* \leqslant \bar{p}^*$, both $f(p)$ and $\bar{f}(p)$ are positive for

$p < p^*$, and negative for $p > \bar{p}^*$. If $0 < s < 1$, $p^* \neq \bar{p}^*$, and $f, \bar{f}$ are not identically zero, $\tilde{p}^*$ is strictly between p^* and $\bar{p}^*$. ∎

5. PRODUCT MEASURES

Now we show that quite a number of edge processes have invariant product measures for their action on *arbitrary graphs*. We consider the continuous time Markov chain which corresponds to the action on a single edge. There are three states $U = 00$, $W = 01$ (including 0) and $Z = 11$. The transition rates are $p_{UW} = 2b$, $p_{UZ} = 2t$, $p_{WU} = d$, $p_{WZ} = g$, $p_{ZU} = 2a$, $p_{ZW} = 2c$. For each edge process, there is a unique stationary distribution (u, w, z) determined by

$$\begin{aligned} 2(b+t)\,u &= dw + 2az \\ (d+g)\,w &= 2bu + 2cz \\ 2(a+c)\,z &= 2tu + gw \end{aligned} \tag{5.1}$$

Proposition 5.1. If there is a $p \in [0, 1]$ such that the above equations are fulfilled with $u = (1-p)^2$, $w = 2p(1-p)$ and $z = p^2$, then the product measure $\mu = \{1-p, p\}^{\mathbf{V}}$ on the configuration space $\{0, 1\}^{\mathbf{V}}$ is a stationary distribution for the given edge process on an arbitrary graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$.

Proof. It suffices to show that for any fixed edge $\{x, y\}$, the action induced by a call of that edge preserves the product measure $\mu(C)$ of any set of the form

$$C = \{\xi \mid \xi(x) = v_x, \xi(y) = v_y, \xi(x_i) = v_i \text{ for } i = 1,.., n\}$$

For given $x_i \in \mathbf{V} \backslash \{x, y\}$ and $v_i \in \{0, 1\}$, however, these cylinder sets are transformed into each other in the same manner as v_x and v_y. Since $\mu(C) = \gamma\beta$ with γ depending on the v_i and $\beta = p^2$, $(1-p)^2$, $p(1-p)$ for $(v_x, v_y) = 11$, 00, 01, and 10, respectively, the above equations show that the flow between the cylinders preserves the values $\mu(C)$. ∎

Let us reformulate the result. In (5.1) the second equation follows from the first and the third. Using $u = (1-p)^2$, $w = 2p(1-p)$, $z = p^2$ we find

Proposition 5.2. The product measure $\mu = \{1-p, p\}^{\mathbf{V}}$ is stationary for the process with parameters a, b, c, d, e, t, g if and only if

$$\begin{aligned} (b+t)(1-p)^2 &= dp(1-p) + ap^2 \\ (a+c)\,p^2 &= gp(1-p) + t(1-p)^2 \end{aligned}$$

With p running from 0 to 1, these equations describe a set of codimension 1 in the simplex $\mathscr{F}$. For legal edge processes ($b = t = 0$) we have necessarily $p = 0$, or $a = d = 0$ and $cp = (1-p)\,g$. Thus the processes with non-trivial stationary product measures in Fig. 2 form the upper triangular face *CGE*, without edges *CE* and *GE*. In Fig. 3 there are no such processes. For spin systems ($a = t = e = 0$) we get $b(1-p) = dp$ and $cp = g(1-p)$ which yields a surface in the tetrahedron BCDG.

The parameter p can be eliminated as follows.

Corollary 5.3. There is a stationary product measure if and only if

$$(cd + da + ag)(bg + gt + td) = (ab + bc + ct)^2$$

Proof. First suppose (5.1) is true. It follows that

$$2(ab + bc + ct)\,u = (cd + da + ag)\,w$$

$$2(ab + bc + ct)\,z = (bg + gt + td)\,w$$

The parameter u, w, z in Proposition 5.1 fulfil $w^2 = 4uz$ which yields (5.3).

Next, we suppose (5.3) is true and derive (5.2). If $ab + bc + ct > 0$, the last equations lead us to define p by $p/(1-p) = \sqrt{z/u} = (bg + gt + td)/(ab + bc + ct) > 0$ which by (5.3) implies $(1-p)/p = (cd + da + ag)/(ab + bc + ct) > 0$. Inserting this into (5.2), equality is easily verified.

If $ab + bc + ct = 0$, there are 3 cases. $b = t = 0$ implies (5.2) with $p = 0$. Similarly, $a = c = 0$ implies (5.2) with $p = 1$. Finally, $b = c = 0$ and $a, t > 0$ reduces (5.2) to $t(1-p)^2 = ap^2$ which holds for a unique $p \in (0, 1)$. ∎

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